AD-A213 536

CMS Technical Summary Report #90-17

SOME RESULTS ON CONNECTING ORBITS FOR A CLASS OF HAMILTONIAN SYSTEMS

Paul H. Rabinowitz
and
Kazunaga Tanaka

UNIVERSITY OF WISCONSIN

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

October 1989

(Received October 17, 1989)

Sponsored by

National Science Foundation
Washington, DC 20550

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Office of Naval Research
Department of the Navy
Arlington, VA 22217

UNIVERSITY OF WISCONSIN - MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

SOME RESULTS ON CONNECTING ORBITS FOR A CLASS OF HAMILTONIAN SYSTEMS

Paul H. Rabinowitz* and Kazunaka Tanaka

CMS Technical Summary Report #90-17

October 1989

Abstract

The existence of various kinds of connecting orbits is established for the Hamiltonian system:

$$(*) \qquad \ddot{q} + V'(q) = 0$$

as well as its time dependent analogue.

For the autonomous case, our main assumption is that V has a global maximum, e.g. at $x = 0$ and we find various kinds of orbits terminating at 0. For the time dependent case V has a local but not global maximum at $x = 0$ and we find a homoclinic orbit emanating from and terminating at 0.

AMS (MOS) Subject Classifications: 34C25, 35J60, 58E05, 58F05, 58F22

Key Words: connecting orbit, Hamiltonian system, periodic solution, homoclinic orbit

*Sponsored in part by the National Science Foundation under Grant # MCS-8110556, the U.S. Army Research Office under contract #DAAL03-87-K-0043, and the Office of Naval Research under Grant No. N00014-88-K-0134. Any reproduction for the purposes of the United States Government is permitted.

Some results on connecting orbits for a class of Hamiltonian Systems

§1. Introduction.

This paper concerns the existence of various kinds of connecting orbits for second order Hamiltonian systems of the form

$$(HS) \qquad \ddot{q} + V'(q) = 0.$$

It will always be assumed that V has a global maximum, e.g. at $x = 0$. Therefore $q \equiv 0$ is a solution of (HS). We are interested in nontrivial solutions of (HS) that terminate at $x = 0$, i.e.

$$(1.1) \qquad \lim_{t \to \infty} q(t) \equiv q(\infty) = 0 = \dot{q}(\infty).$$

If e.g. $\Omega \subset \mathbf{R}^n$ is bounded, $V \in C^1(\overline{\Omega}, \mathbf{R})$ and

$$(1.2) \qquad V(x) < V(0) \quad \text{for all} \quad x \in \overline{\Omega} \setminus \{0\},$$

Theorem 2.1 shows there is a solution of (HS) such that $q(0) \in \partial\Omega$, $q(t) \in \Omega$ for all $t > 0$, and (1.1) holds. If (1.2) is replaced by

$$(1.3) \qquad \begin{cases} V(x) < V(0) & \text{for all } x \in \Omega \setminus \{0\} \\ V(x) = V(0) & \text{for } x \in \partial\Omega \\ V'(x) \neq 0 & \text{for } x \in \partial\Omega, \end{cases}$$

then Theorem 2.22 shows there exists a solution of (HS) as in Theorem 2.1 further satisfying $\dot{q}(0) = 0$. Since (HS) is time reversible, extending this solution to $\mathbf{R}$ via $q(-t) = q(t)$ yields a homoclinic solution of (HS) emanating from 0.

These results are proved in §2 by studying the functional

$$(1.4) \qquad I(q) = \int_0^\infty \left[\frac{1}{2} |\dot{q}(t)|^2 - V(q) \right] dt$$

corresponding to (HS) and using elementary minimization arguments. Our arguments were motivated in part by [1] where related reasoning was employed to prove the existence of heteroclinic solutions of (HS) for functions $V(x)$ which are periodic in x.

In §3 the setting of Theorem 2.22 is studied in more detail. If $\partial\Omega$ contains ℓ components, we show in Theorem 3.1 that (HS) possesses at least ℓ_h homoclinic orbits joining 0 to $\partial\Omega$ and at least ℓ_p periodic solutions joining the components, where $\ell_h + \ell_p \geq \ell$.

In §4, we assume $\Omega = \mathbf{R}^n$. Then the methods of §2 show that for each $\xi \in \mathbf{R}^n \backslash \{0\}$, there is a solution of (HS) satisfying $q(0) = \xi$ and (1.1) (Theorem 4.1). Moreover the critical value of I in (1.4) has a variational characterization. The existence assertion of Theorem 4.1 without this variational characterization was already proved by Bolotin and Kozlov [2] in a more general setting by a less direct argument. It is also proved in §4 that under the hypotheses of Theorem 4.1, there is a solution of (HS) emanating from infinity and terminating at 0.

Lastly in §5, the existence of homoclinic orbits is studied for

$$\ddot{q} + V_q(t, q) = 0 \tag{1.5}$$

assuming that the potential energy V grows at a superquadratic rate as $|x| \to \infty$, i.e.

$$V(t, x)|x|^{-2} \to \infty \qquad \text{as} \quad |x| \to \infty. \tag{1.6}$$

Such questions have been studied recently by Coti-Zelati, Ekeland, and Sere [3], and Hofer and Wysocki [4] for general Hamiltonian systems:

$$\dot{z} = \mathcal{J} H_z(t, z) \tag{1.7}$$

and for (1.5) in [5]. It is assumed in [3-5] that H (or V) is T-periodic in t. Here (1.5) is treated without such an assumption for

$$V(t, x) = -\frac{1}{2} L(t) x \cdot x + W(t, x) \tag{1.8}$$

where $L(\cdot)$ is positive definite and W satisfies (1.6). Two different settings are studied. For the first, it is assumed that the smallest eigenvalue of $L(t)$ approaches ∞ as $|t| \to \infty$. For the second, we assume

$$L(t) \to L_\infty(t),$$

a T-periodic function, in an appropriate sense as $|t| \to \infty$. For both cases minimax arguments are employed to obtain the existence of the homoclinic solution.

This work was done while the second author was visiting the Center for the Mathematical Sciences, University of Wisconsin – Madison. He would like to thank the Center for its kind hospitality.

§2. Autonomous Hamiltonian Systems

This section contains Theorem 2.1 and 2.22 as stated in the Introduction. In this and the sections that follow, in the proofs of the results, it will be assumed that $n \geq 2$. The proofs for $n = 1$ are much simpler.

To begin we have:

Theorem 2.1: Let Ω be a bounded neighborhood of 0 in $\mathbf{R}^n$ and $V \in C^1(\overline{\Omega}, \mathbf{R})$ with $V(x) < V(0)$ for all $x \in \overline{\Omega}\backslash\{0\}$. Then there exists a solution, q, of (HS) such that $q(0) \in \partial\Omega$, $q(\infty) = 0 = \dot{q}(\infty)$, and $q(t) \in \Omega$ for all $t \in (0, \infty)$.

Proof: Without loss of generality, it can be assumed that $V(0) = 0$. The proof consists of several steps and follows the lines of a related situation in [1] involving the existence of heteroclinic orbits of details.

Let $\mathbf{R}^+ = [0, \infty)$ and

$$E = \{q \in W^{1,2}_{\text{loc}}(\mathbf{R}^+, \mathbf{R}^n) \mid \int_0^\infty |\dot{q}|^2 dt < \infty\}. \tag{2.2}$$

E is a Hilbert space under the norm

$$\|q\|^2 = \int_0^\infty |\dot{q}|^2 dt + |q(0)|^2$$

and $E \subset C(\mathbf{R}^+, \mathbf{R}^n)$. Let Γ be the subset of E defined by

$$\Gamma = \{q \in E \mid q(0) \in \partial\Omega, q(\infty) = 0, \text{ and } q(t) \in \overline{\Omega} \text{ for all } t \in \mathbf{R}^+\}. \tag{2.3}$$

For $q \in \Gamma$, consider the functional

$$I(q) = \int_0^\infty \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt. \tag{2.4}$$

Set

$$c = \inf_{q \in \Gamma} I(q). \tag{2.5}$$

Then Theorem 2.1 follows on establishing:

Proposition 2.6: c is a critical value of I with a corresponding critical point $q \in \hat{\Gamma}$ where

$$\hat{\Gamma} = \{w \in \Gamma \mid w(t) \in \Omega \text{ for all } t \in (0, \infty)\}. \tag{2.7}$$

Moreover

$$c = \inf_{w \in \Gamma} I(w), \tag{2.8}$$

q satisfies (HS), and $\dot{q}(\infty) = 0$.

The first step in proving Proposition 2.6 is a variant of Lemma 3.6 of [1]. Let $B_\rho(\xi)$ denote the open ball of radius ρ about $\xi \in \mathbf{R}^n$. If $\xi = 0$, we simply write B_ρ.

Lemma 2.9: Let $\rho > 0$ such that $B_\rho \subset \overline{\Omega}$. Set

$$\beta(\rho) = \min_{x \in \overline{\Omega} \setminus B_\rho} -V(x).$$

Suppose $w \in E$ and $w(t) \in \overline{\Omega} \setminus B_\rho$ for $t \in \bigcup_{j=1}^k [r_j, s_j]$. Then

$$I(w) \geq \sqrt{2\beta(\rho)} \sum_{j=1}^k |w(r_j) - w(s_j)|. \tag{2.10}$$

Proof: The proof of lemma 2.9 is same as its analogue in [1] so we refer to [1] for details.

An immediate consequence of Lemma 2.9 is

Lemma 2.11: If $w \in E$, $w(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}^+$, and $I(w) < \infty$, then $w(\infty) = 0$. If in addition, $w(0) \in \partial\Omega$, then $w \in \Gamma$.

Proof: Let $\omega(w)$ denote the set of limit points of the orbit $w(t)$ as $t \to \infty$. Since $\overline{\Omega}$ is compact, $\omega(w) \neq \phi$. Let $\xi \in \omega(w)$. If $\xi \neq w(\infty)$, there is a $\delta > 0$ and sequences $(t_m), (\tau_m) \subset \mathbf{R}^+$ such that $t_m \to \infty$, $w(t_m) \to \xi$, $\tau_m \to \infty$, and $w(\tau_m) \notin B_\delta(\xi)$ as $m \to \infty$. Applying Lemma 2.9 with $\rho = \delta/4$ shows $I(w) \geq k\sqrt{2\beta(\delta/4)}$ for any $k \in \mathbf{N}$, i.e. $I(w) = \infty$, a contradiction. Therefore $\xi = w(\infty)$. Lastly observe that since $V(0) > V(x)$ for all $x \in \overline{\Omega} \setminus \{0\}$, the only possible value of ξ for which $I(w) < \infty$ is $\xi = 0$.

Now we can prove:

Proposition 2.12: There exists $q \in \Gamma$ such that $I(q) = c$.

Proof: Let (q_m) be a minimizing sequence for (2.5). Since $V \leq 0$ and $\overline{\Omega}$ is compact, the form of I shows (q_m) is bounded in E. Hence a subsequence of (q_m) converges weakly in E and strongly in $L^\infty_{\text{loc}}(\mathbf{R}^+, \mathbf{R}^n)$ to $q \in E$ satisfying $q(0) \in \partial\Omega$ and $q(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}^+$. A simple lower semicontinuity argument – see Proposition 3.12 of [1] – shows $I(q) < \infty$ and

$$I(q) \leq \inf_{w \in \Gamma} I(w). \tag{2.13}$$

Consequently $q \in \Gamma$ via Lemma 2.11 and equality holds in (2.13).

To complete the proof of Proposition 2.6 and Theorem 2.1, it remains to show that $q \in \hat{\Gamma}$, i.e. $q(t) \in \Omega$ for all $t \in (0, \infty)$, q satisfies (HS), and $\dot{q}(\infty) = 0$. Set

$$\mathcal{T} = \{t \in (0, \infty) \mid q(t) \in \partial\Omega\}.$$

Since $q(\infty) = 0$ and q is continuous, $\mathcal{T}$ is compact.

Proposition 2.14: $\mathcal{T} = \phi$ and $q \in \hat{\Gamma}$.

Proof: If $\mathcal{T} \neq \phi$, define τ by

$$\tau \equiv \max\{t \in (0, \infty) \mid t \in \mathcal{T}\}. \tag{2.15}$$

Hence $\tau > 0$ and since $q(\infty) = 0$, $\tau < \infty$. Set $Q(t) = q(t + \tau)$. Then $Q \in \hat{\Gamma} \subset \Gamma$ and

$$I(Q) = \int_{\tau}^{\infty} \left[\frac{1}{2}|\dot{q}|^2 - V(q(t))\right] dt < I(q) \tag{2.16}$$

since $V(q(t)) \leq 0$ and $\not\equiv 0$ for $t \in (0, \tau)$. But then (2.13) is violated. Hence $\mathcal{T} = \phi$ and $q \in \hat{\Gamma}$.

Proposition 2.17: q satisfies (HS) on $(0, \infty)$.

Proof: The proof of this statement is identical to that of Proposition 3.18 of [1] and will be omitted.

The Proposition implies

Corollary 2.18: $\dot{q}(\infty) = 0$.

Proof: Since q satisfies (HS), there is a constant A such that

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) \equiv A \tag{2.19}$$

for all $t \in \mathbf{R}^+$. Consequently

$$I(q) = \int_0^{\infty} (|\dot{q}|^2 - A)dt. \tag{2.20}$$

Since $q \in E$ and $I(q) < \infty$, (2.20) shows $A = 0$. Finally $q(\infty) = 0$, $V(0) = 0$, and (2.19) imply that $\dot{q}(\infty) = 0$.

The proof of Theorem 2.1 is complete.

Remark 2.21: If V merely belongs to $C^1(\overline{\Omega}, \mathbf{R})$, the initial value problem for (HS) need not have a unique solution. Therefore it is possible that the solution $q(t)$ just constructed satisfies $q(t) \equiv 0$ for large t. Of course this cannot happen if $V \in C^2(\overline{\Omega}, \mathbf{R})$.

As a consequence of Theorem 2.1, we have

Theorem 2.22: Let $V \in C^1(\mathbf{R}^n, \mathbf{R})$ with $V(0) = 0$ being a strict local maximum of V. Let

$$V^0 = \{x \in \mathbf{R}^n \mid V(x) < 0\} \cup \{0\}$$

and let Ω be the component of V^0 containing 0. If Ω is bounded and $V'(x) \neq 0$ for all $x \in \partial\Omega$, then there is a solution, q, of (HS) in $\hat{\Gamma}$ with $\dot{q}(0) = 0 = \dot{q}(\infty)$. Moreover $I(q) = c$ where c is given by (2.5) and (2.8).

Proof: An approximation argument based on Theorem 2.1 will be employed. For $\epsilon > 0$, let

$$\Omega_\epsilon = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \epsilon\}.$$

For small ϵ, Ω_ϵ is a neighborhood of 0 for which Theorem 2.1 is valid. Hence there exists a solution q_ϵ of (HS) such that if

$$\Gamma_\epsilon = \{w \in E \mid w(0) \in \partial\Omega_\epsilon, w(\infty) = 0, \text{ and } \\ w(t) \in \overline{\Omega}_\epsilon \text{ for all } t \in (0, \infty)\},$$

and

$$\hat{\Gamma}_\epsilon = \{w \in E \mid w(0) \in \partial\Omega_\epsilon, w(\infty) = 0, \text{ and } \\ w(t) \in \Omega_\epsilon \text{ for all } t \in (0, \infty)\},$$

then

$$\text{(2.23)} \qquad \begin{cases} (i) \ \ q_\epsilon \in \hat{\Gamma}_\epsilon \ \text{ and } \ \dot{q}_\epsilon(\infty) = 0 \\ (ii) \ \ I(q_\epsilon) = \inf_{\hat{\Gamma}_\epsilon} I = \inf_{\Gamma_\epsilon} I \equiv c_\epsilon \end{cases} .$$

Since $\Omega_\epsilon \subset \Omega$ which is bounded and q_ϵ is a solution of (HS), $\{q_\epsilon\}$ is bounded in $C^2(\mathbf{R}^+, \mathbf{R}^n)$. Hence there is a subsequence $\epsilon_k \to 0$ and $q \in C^2(\mathbf{R}^+, \mathbf{R}^n)$ such that $q_{\epsilon_k} \to q$ in $C^2_{\mathrm{loc}}(\mathbf{R}^+, \mathbf{R}^n)$. By (HS) and (2.23), q is a solution of (HS), $q(0) \in \partial\Omega$, $q(t) \in \overline{\Omega}$ for all $t \in (0, \infty)$, and

$$\text{(2.24)} \qquad \frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = 0 \quad \text{for } t \in \mathbf{R}^+.$$

Since $q(0) \in \partial V^0$, (2.24) shows $\dot{q}(0) = 0$. It remains to show that $q \in \hat{\Gamma}$, $\dot{q}(\infty) = 0$, and (2.5) and (2.8) hold. These facts are consequences of the next 3 lemmas. For now let c be as given by (2.5) and $\hat{c}$ by (2.8) so

$$\text{(2.25)} \qquad c \leq \hat{c}.$$

Lemma 2.26: $q \in E$ and $I(q) \leq c$.

Proof: Note that for $0 \leq \epsilon < \delta$, if $p \in \Gamma_\epsilon$, since $p(\infty) = 0$, there exists a $\tau > 0$ such that $p(\tau) \in \partial\Omega_\delta$ and $p(t) \in \Omega_\delta$ for all $t > \tau$. Hence $w(t) = p(t + \tau) \in \Gamma_\delta$ and $I(w) \leq I(p)$. This implies that for such pairs p and w:

$$\inf_{p \in \Gamma_\epsilon} I(p) \geq \inf_{w \in \Gamma_\delta} I(w) \geq \inf_{u \in \Gamma_\delta} I(u) \tag{2.27}$$

for $0 \leq \epsilon < \delta$ and therefore

$$\hat{c} \geq c \geq c_\epsilon \geq c_\delta. \tag{2.28}$$

Since $q_{\epsilon_k} \to q$ in C^2_{loc}, for all $T > 0$,

$$\begin{aligned} \infty > \hat{c} \geq c &\geq \overline{\lim}_{k \to \infty} I(q_{\epsilon_k}) \\ &\geq \overline{\lim}_{k \to \infty} \int_0^T \left[\frac{1}{2}|\dot{q}_{\epsilon_k}|^2 - V(q_{\epsilon_k})\right] dt \\ &= \int_0^T \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt. \end{aligned} \tag{2.29}$$

T being arbitrary, (2.29) shows $q \in E$ and $I(q) \leq c$.

Lemma 2.30: $q(\infty) = 0 = \dot{q}(\infty)$.

Proof: If $q(\infty) = 0$, by (2.24), $\dot{q}(\infty) = 0$. To prove that $q(\infty) = 0$, an indirect argument is employed. If $q(\infty) \neq 0$, as in the proof of Lemma 2.11, there is a sequence $t_m \to \infty$ and $\xi \neq 0$ such that $q(t_m) \to \xi$ as $m \to \infty$. A slight modification of the argument of Lemma 2.11 shows that $V(\xi) = 0$, i.e. $\xi \in \partial\Omega$ and therefore $V'(\xi) \neq 0$. Consequently there is a neighborhood U of ξ in $\mathbf{R}^n$ such that

$$|V'(x) - V'(\xi)| \leq \frac{1}{2}|V'(\xi)| \tag{2.31}$$

for all $x \in U$. Since by (2.24), $\dot{q}$ is bounded in L^∞, there is a $\delta > 0$ and $m_0 \in \mathbf{N}$ such that

$$q\left(\bigcup_{m \geq m_0} [t_m, t_m + \delta]\right) \subset U. \tag{2.32}$$

Hence for $t \in (t_m, t_m + \delta)$, by (2.31),

$$\begin{aligned} |\dot{q}(t) - \dot{q}(t_m)| &= \left|\int_{t_m}^t V'(q(s))ds\right| \\ &= \left|(t - t_m)V'(\xi) - \int_{t_m}^t (V'(\xi) - V'(q(s)))ds\right| \\ &\geq \frac{1}{2}(t - t_m)|V'(\xi)| \end{aligned} \tag{2.33}$$

and

$$\int_{t_m}^{t_m+\delta} |\dot{q}(s)|^2 ds \geq \frac{1}{\delta}\left(\int_{t_m}^{t_m+\delta} |\dot{q}(s)| ds\right)^2 \tag{2.34}$$

$$\geq \frac{1}{\delta}\left(\frac{\delta^2}{4}|V'(\xi)| - \delta|\dot{q}(t_m)|\right)^2.$$

By (2.24), $\dot{q}(t_m) \to 0$ as $m \to \infty$. Hence by (2.34),

$$I(q) \geq \frac{1}{2}\sum_{m=m_0}^{\infty}\int_{t_m}^{t_m+\delta} |\dot{q}(t)|^2 dt = \infty, \tag{2.35}$$

contrary to Lemma 2.26. Hence $\xi = 0$ and $q(\infty) = 0$.

Note that by Lemmas 2.26 and 2.30, $q \in \Gamma$.

Lemma 2.36: q satisfies (2.5) and (2.8) and $q(t) \in \Omega$ for all $t \in (0, \infty)$.

Proof: Lemma 2.26 shows $I(q) \leq c \leq \hat{c}$. Since $q \in \Gamma$, (2.5) holds. To verify (2.8), it suffices to prove that $q(t) \in \Omega$ for all $t \in (0, \infty)$ and therefore $q \in \hat{\Gamma}$. Let

$$\mathcal{T} = \{t \in (0, \infty) \mid q(t) \in \partial\Omega\}.$$

By Lemma 2.30, $\mathcal{T}$ is a bounded set. If $\mathcal{T} \neq \phi$, set $\tau = \max\{t \mid t \in \mathcal{T}\}$ and $Q(t) = q(t+\tau)$. Then $Q \in \hat{\Gamma}$ and $I(Q) \leq I(q)$ with equality holding if and only if

$$q(t) \equiv q(0) \in \partial\Omega \quad \text{for } t \in [0, \tau]. \tag{2.37}$$

But $q(t)$ is a solution of (HS) for all $t > 0$ and by hypothesis $V'(x) \neq 0$ on $\partial\Omega$. Therefore (2.37) cannot hold. Hence

$$\hat{c} \leq I(Q) < I(q) = c, \tag{2.38}$$

contrary to (2.25).

The proof of Theorem 2.22 is complete. As an immediate consequence of the Theorem we have:

Corollary 2.39: Under the hypotheses of Theorme 2.22, (HS) possesses a homoclinic solution q with $q(\pm\infty) = 0 = \dot{q}(\pm\infty)$ and $q(0) \in \partial\Omega$, $\dot{q}(0) = 0$.

Proof: Observing that (HS) is time reversible, the solution q obtained in Theorem 2.22 can be extended to $\mathbf{R}$ via $q(-t) = q(t)$. Since $\dot{q}(0) = 0$, this extension furnishes a solution of the desired type.

Remark 2.40: By combining the ideas of Theorem 2.1 and 2.22, one can prove a stronger version of Theorem 2.1 where the assumption that $V(x) < V(0)$ for all $x \in \overline{\Omega}\backslash\{0\}$ is replaced by $V(x) < V(0)$ for all $x \in \Omega\backslash\{0\}$ and $V(x) = V(0)$ implies $V'(x) \neq 0$ for $x \in \partial\Omega$.

Remark 2.41: In Theorem 2.22, the assumption that $V'(x) \neq 0$ for all $x \in \partial\Omega$ can be eliminated. However then the conclusion becomes: either there exists a homoclinic orbit joining 0 to $\partial\Omega$ (as in Corollary 2.39) or there is an orbit of heteroclinic type joining 0 and $\partial\Omega$, i.e. there exists a $T^* \geq -\infty$ and a solution q of (HS) with $q \in C^2((T^*, \infty), \mathbf{R}^n)$, $q(\infty) = \dot{q}(\infty) = 0$ and

$$\text{dist}(q(t), \partial\Omega) \to 0 \quad \text{as } t \to T^*. \tag{2.42}$$

(If $V' = 0$ at only finitely many points on $\partial\Omega$, (2.42) simplifies to $q(t)$ approaches one of these points as $t \to T^*$). We will not carry out the details of the proof, which is based on the arguments of Theorem 2.22 and results from [1]. However we will give a quick sketch. Arguing as in the proof of Theorem 2.22, q_{ϵ_k} converges in C^2_{loc} to $q \in C^2$, a solution of (HS) with $q(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}^+$, $q(0) \in \partial\Omega$, and (2.24) holds. If the functions $q_\epsilon(t)$ avoid a neighborhood of the set

$$S = \{x \in \partial\Omega \mid V'(x) = 0\},$$

the arguments of Theorem 2.22 and Corollary 2.39 carry over to show q is a homoclinic orbit of (HS). If however,

$$\text{dist}(q_\epsilon(t), S) \to 0,$$

by appropriately rescaling time and using (2.23) (ii) and arguments from [1], a subsequence of (q_{ϵ_k}) will converge to a heteroclinic or homoclinic orbit of (HS) emanating from 0. See also the proof of Theorem 3.1 for related rescaling arguments.

§3. A refined version of Theorem 2.22.

Returning to the setting of Theorem 2.22, note that $\partial\Omega$ can have at most finitely many components, for otherwise there would exist an accumulation point z of these components. Consequently $z \in \partial\Omega$ and $V(z) = 0$. Hence $V'(z) \neq 0$. But then the Implicit Function Theorem shows there is only one component of $\partial\Omega$ near z. Let $S_1, \dots, S_\ell$ denote the components of $\partial\Omega$. Our main result in this section, Theorem 3.1, refines Theorem 2.22 and gives a lower bound for the number of "periodic" and "homoclinic" orbits in the sense of Corollary 2.39, i.e. the homoclinic orbits begin at 0 at $t = -\infty$ bounce off of some S_j at $t = 0$ and return to 0 at $t = \infty$. Likewise the periodic orbits bounce back and forth between a pair of components of $\partial\Omega$ in finite time.

A result due to Bolotin and Kozlov, related to Theorem 3.1, but when there are no homoclinic orbits present and involving a rather different proof can be found in [2, Theorem 3].

Theorem 3.1. Under the hypotheses of Theorem 2.22, let ℓ_h be the number of homoclinic orbits of (HS) emanating from 0 and let ℓ_p be the number of periodic (after reflection) orbits of (HS) which have energy 0 and which join components of $\partial\Omega$. Then $\ell_k + \ell_p \geq \ell$.

Proof: Note that Γ is the union of ℓ components:

$$\Gamma_i = \{q \in E \mid q(0) \in S_i, q(\infty) = 0, \text{ and } q(t) \in \overline{\Omega} \text{ for all } t \in \mathbf{R}^+\}, \quad 1 \leq i \leq \ell. \tag{3.2}$$

Similarly for ϵ small, Γ_ϵ consists of ℓ components:

$$\Gamma_{i,\epsilon} = \{q \in E \mid q(0) \in S_{i,\epsilon}, q(\infty) = 0, \text{ and } q(t) \in \overline{\Omega}_\epsilon \text{ for all } t \in \mathbf{R}^+\}, \quad 1 \leq i \leq \ell. \tag{3.3}$$

where

$$S_{i,\epsilon} = \{x \in \Omega \mid \operatorname{dist}(x, S_i) = \epsilon\}, \quad 1 \leq i \leq \ell. \tag{3.4}$$

Sets $\hat{\Gamma}_i, \hat{\Gamma}_{i,\epsilon}$ can be defined similarly.

Let

$$c_i = \inf_{q \in \Gamma_i} I(q), \quad 1 \leq i \leq \ell \tag{3.5}$$

and

$$c_{i,\epsilon} = \inf_{q \in \Gamma_{i,\epsilon}} I(q), \quad 1 \le i \le \ell. \tag{3.6}$$

We fix i and consider $c_{i,\epsilon}$. It follows from the proof of Theorem 2.1 that there is a $q_\epsilon \in \Gamma_i$ such that $I(q_\epsilon) = c_{i,\epsilon}$. Moreover by the argument of Proposition 3.18 of [1], q_ϵ is a solution of (HS) for $t \in (0, \infty) \backslash \mathcal{T}_\epsilon$ where

$$\mathcal{T}_\epsilon = \{t \in (0, \infty) \mid q_\epsilon(t) \in \bigcup_{r=1}^{\ell} S_{r,\epsilon}\}. \tag{3.7}$$

Two possibilities occur: (i) $\mathcal{T}_\epsilon \neq \phi$ for all small $\epsilon > 0$ or (ii) $\mathcal{T}_\epsilon = \phi$ for a sequence $\epsilon_m \to 0$.

Case (i): $\mathcal{T}_\epsilon \neq \phi$ for all small $\epsilon > 0$.

This is the more complicated situation. We will show that there is a "chain" which consists of pieces of periodic and homoclinic orbits of (HS) and which joins S_i and 0. More precisely we have:

Proposition 3.8: If $\mathcal{T}_\epsilon \neq \phi$ for all small $\epsilon > 0$, there is a $p = p(i) \le \ell$ and a map $j : \{1, \dots, p\} \to \{1, \dots, \ell\}$ where $j(1) = i$ and $j(s) \neq j(s')$ if $s \neq s'$, numbers $T_1, \dots, T_p > 0$. and functions $Q_1, \dots, Q_{p+1}$ such that:

(a) Q_s is a $2T_s$ periodic solution of (HS), $1 \le s \le p$.

(b) $Q_s(0) \in S_{j(s)}$, $Q_s(T_s) \in S_{j(s+1)}$, $\dot{Q}_s(0) = 0 = \dot{Q}_s(T_s)$, $1 \le s \le p$.

(c) $Q_s(t) \in \Omega$ for all $t \in (0, T_s)$.

(d) Q_{p+1} is a "homoclinic" orbit of (HS) such that $Q_{p+1} \in \hat{\Gamma}_{j(p)}$ and $\dot{Q}_{p+1}(0) = 0$.

(e) Setting $T_{p+1} = \infty$,

$$\sum_{s=1}^{p+1} \int_0^{T_s} \left[\frac{1}{2}|\dot{Q}_s|^2 - V(Q_s)\right] dt = \inf_{\Gamma_i} I = \inf_{\hat{\Gamma}_i} I,$$

(f) $\frac{1}{2}|\dot{Q}_s(t)|^2 +$ [illegible] $_s(t)) = 0$ on $[0, T_s]$, $1 \le s \le p+1$.

Proof: Since $\mathcal{T}_\epsilon \neq$ [illegible] all small $\epsilon > 0$, numbers

$$\sigma_1^\epsilon \ge \tau_2^\epsilon > \sigma_2^\epsilon \ge \cdots > \sigma_{p+1}^\epsilon = 0 \tag{3.9}$$

can be defined as follows:

$$\sigma_1^\epsilon = \sup\{t \in (0, \infty) \mid q_\epsilon(t) \in \bigcup_r S_{r,\epsilon}\}. \tag{3.10}$$

Suppose $q_\epsilon(\sigma_1^\epsilon) \in S_{r_1,\epsilon}$, r_1 depending on ϵ. Clearly $r_1 \neq i$ via (3.6). Set

$$\tau_2^\epsilon = \inf\{t \in [0, \sigma_1^\epsilon] \mid q_\epsilon(t) \in S_{r_1,\epsilon}\} \tag{3.11}$$

so $\tau_2^\epsilon \in (0, \sigma_1^\epsilon]$. Set

$$\sigma_2^\epsilon = \sup\{t \in [0, \tau_2^\epsilon) \mid q_\epsilon(t) \in \bigcup_r S_{r,\epsilon}\}. \tag{3.12}$$

If $\sigma_2^\epsilon = 0$, $p = 1$ and we are through. If not, say $q_\epsilon(\sigma_2^\epsilon) \in S_{r_2,\epsilon}$. Clearly $r_2 \neq i$ or r_1. Set

$$\tau_3^\epsilon = \inf\{t \in [0, \sigma_2^\epsilon] \mid q_\epsilon(t) \in S_{r_2,\epsilon}\}. \tag{3.13}$$

Continuing in this fashion, in at most ℓ steps, we find the σ's and τ's as in (3.9). Note that p and the indices r_k depend on ϵ as well as i and $r_{p+1} = i$. However since only at most $\ell - 1$ sets are involved, a subsequence $\epsilon_m \to 0$ can be chosen so that p and $r_1, \ldots$ are independent of ϵ. We set $j(s) = r_{p-s+1}$.

Proposition 3.14: For ϵ sufficiently small,

(i) There is a $\delta > 0$ (independent of ϵ) such that $q_\epsilon(t) \notin B_\delta(0)$ for all $t \in \bigcup_{s=2}^{j+1} (\sigma_s^\epsilon, \tau_s^\epsilon)$,

(ii) $q_\epsilon(t)$ is a solution of (HS) on $(\sigma_s^\epsilon, \tau_s^\epsilon)$ for each $s \in \{2, \ldots, p+1\}$,

(iii) There is an $M > 0$ (independent of ϵ) such that $|\sigma_s^\epsilon - \tau_s^\epsilon| \leq M$ for $s \in \{2, \ldots, p+1\}$.

Proof. If (i) is not true, for some $s \in \{2, \ldots, p+1\}$ there is a sequence $\epsilon_m \to 0$ and $\mu_s^\epsilon \in (\sigma_s^\epsilon, \tau_s^\epsilon)$ (with $\epsilon = \epsilon_m$) such that $q_\epsilon(\mu_s^\epsilon) \to 0$ as $m \to \infty$. Set

$$Q_\epsilon(t) = \begin{cases} q_\epsilon(t), & t \in (0, \mu_s^\epsilon) \\ (\mu_s^\epsilon + 1 - t) q_\epsilon(\mu_s^\epsilon), & t \in [\mu_s^\epsilon, \mu_s^\epsilon + 1] \\ 0 & t \geq \mu_s^\epsilon + 1\ . \end{cases} \tag{3.15}$$

Then $Q_\epsilon(t) \in \Gamma_{i,\epsilon}$ and

$$\begin{aligned} I(Q_\epsilon) - I(q_\epsilon) = &\int_{\mu_s^\epsilon}^{\mu_s^\epsilon + 1} \left[\frac{1}{2}|q_\epsilon(\mu_s^\epsilon)|^2 - V(Q_\epsilon)\right] dt \\ &- \int_{\mu_s^\epsilon}^{\infty} \left[\frac{1}{2}|\dot{q}_\epsilon(t)|^2 - V(q_\epsilon(t))\right] dt. \end{aligned} \tag{3.16}$$

The first integral on the right hand side of (3.16) approaches 0 as $m \to \infty$ while Lemma 2.9 shows the second term is bounded away from 0. Hence $I(Q_{\epsilon_m}) < I(q_{\epsilon_m})$ for m large, contrary to (3.6). To prove (ii), it suffices to show that $\mathcal{T}_\epsilon \cap (\sigma_s^\epsilon, \tau_s^\epsilon) = \phi$. But this is

immediate from the definition of the σ's and τ's and (3.6). Lastly to get (iii), suppose that $\tau_s^\epsilon - \sigma_s^\epsilon \to \infty$ along some sequence $\epsilon_m \to 0$. Then by earlier arguments $q_{\epsilon_m}(t - \sigma_s^{\epsilon_m})$ converges in $C^2_{\text{loc}}(\mathbf{R}^+, \mathbf{R}^n)$ to $\tilde{q} \in C^2(\mathbf{R}^+, \mathbf{R}^n)$ such that $\tilde{q}$ satisfies (HS), $\tilde{q}(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}^+$, $\tilde{q}(0) \in S_{r_s}$, and

$$I(\tilde{q}) \leq \varliminf_{m\to\infty} I(q_{\epsilon_m}) < \infty. \tag{3.17}$$

The argument of Lemma 2.30 shows $\tilde{q}(\infty) = 0$. But $q_\epsilon(t) \notin B_\delta(0)$ for all $t \in (\sigma_s^\epsilon, \tau_s^\epsilon)$ via (i) so this is impossible and there exists M as stated.

Now the functions Q_s can be constructed. Letting $m \to \infty$, by Proposition 3.14 (iii), a subsequence ϵ_m can be chosen so that

$$\lim_{m\to\infty} \tau_s^{\epsilon_m} - \sigma_s^{\epsilon_m}$$

exists. Denoting this limit by T_{p-s+2}, as above along a subsequence,

$$\begin{aligned}
q_{\epsilon_m}(t - \sigma_1^{\epsilon_m}) &\longrightarrow Q_{p+1}(t) \text{ in } C^2_{\text{loc}}(\mathbf{R}^+, \mathbf{R}^n),\\
q_{\epsilon_m}(t - \sigma_2^{\epsilon_m}) &\longrightarrow Q_p(t) \text{ in } C^2([0, T_p], \mathbf{R}^n),\\
&\vdots\\
q_{\epsilon_m}(t - \sigma_{p+1}^{\epsilon_m}) &\longrightarrow Q_1(t) \text{ in } C^2([0, T_1], \mathbf{R}^n).
\end{aligned}$$

By Proposition 3.14 (ii) and our construction, Q_s is a solution of (HS) on $(0, T_s)$ with $Q_s(0) \in S_{j(s)}$, $Q_s(T_s) \in S_{j(s+1)}$. Moreover weaker forms of (c) and (d) of Proposition 3.8 hold:

$$\begin{cases} (\bar{c}) \ \ Q_s(t) \in \overline{\Omega} \text{ for all } t \in [0, T_s], \\ (\bar{d}) \ \ Q_{p+1} \text{ is a “homoclinic” orbit of (HS)} \\ \qquad \text{such that } Q_{p+1} \in \Gamma_{j(p)} \text{ and } \dot{Q}_{p+1}(0) = 0. \end{cases} \tag{3.18}$$

Assume (e) and (f) of Proposition 3.8 for the moment. Then the remainder of (b) follows and extending Q_s as an even function about 0 and T_s.

Proof of (e): Clearly

$$\begin{aligned}
&\sum_{s=1}^{p+1} \int_0^{T_s} \left[\frac{1}{2}|\dot{Q}_s|^2 - V(Q_s)\right] dt \\
&\leq \varliminf_{m\to\infty} \sum_{s=1}^{p+1} \int_{\sigma_s^{\epsilon_m}}^{\tau_s^{\epsilon_m}} \left[\frac{1}{2}|\dot{q}_{\epsilon_m}|^2 - V(q_{\epsilon_m})\right] dt \\
&\leq \varliminf_{m\to\infty} I(q_{\epsilon_m}) = \lim_{\epsilon\to 0} \inf_{w\in\Gamma_{i,\epsilon}} I(w) \leq \inf_{w\in\Gamma_i} I(w),
\end{aligned} \tag{3.19}$$

the last inequality following as in (2.28). To show that equality holds in (3.19), note first that since $\partial\Omega$ is C^1, for any $w \in \Gamma_i$ with $I(w) < \infty$, there exists a sequence $w_k \in \hat{\Gamma}_i$ such that $w_k \to w$ in E and $I(w_k) \to I(w)$. Therefore it suffices to find a family of functions $(\eta_r(t)) \subset \Gamma_i$ such that

$$(3.20) \qquad I(\eta_r) \to \sum_{s=1}^{p+1} \int_0^{T_s} \left[\frac{1}{2}|\dot{Q}_s|^2 - V(Q_s)\right] dt$$

as $r \to \infty$. Let $\nu_s \in C^1([0,1], S_{j(s+1)})$ such that $\nu_s(0) = Q_s(T_s)$ and $\nu_s(1) = Q_{s+1}(0)$. Let $\nu_{s,r} = \nu_s(t/r) \in C^1([0,r], S_{j(s+1)})$. Finally let $\eta_r(t)$ be the curve obtained in a natural way by joining the functions $Q_1(t), \nu_{1,r}(t), Q_2(t), \nu_{2,r}(t), \ldots, Q_{p+1}(t)$. Then $\eta_r \in \Gamma_i$. Since

$$(3.21) \qquad \int_0^r \left[\frac{1}{2}|\dot{\nu}_{r,s}|^2 - V(\nu_{r,s})\right] dt = \frac{1}{2r^2}\int_0^r |\dot{\nu}_s(t/r)|^2 dt \to 0$$

as $r \to \infty$, (3.20) follows and (e) is proved.

Proof of (f): Since as in Lemma 2.30, $Q_{p+1}(\infty) = 0 = \dot{Q}_{p+1}(\infty)$, (f) holds for $s = p+1$. Thus suppose $1 \le s \le p$. Since by the part of (b) already established, $V(Q_s(0)) = 0$, it suffices to prove $\dot{Q}_s(0) = 0$. If not, a function $w \in \Gamma_i$ will be constructed such that

$$(3.22) \qquad I(w) < \inf_{\Gamma_i} I.$$

But (3.22) is contrary to (e) so (f) must hold. To find w, let $a > 0$ be small and $L > 1$. Set

$$(3.23) \qquad \xi(t) = Q_s(\frac{t}{L}), \quad t \in (0, La)$$

$$= Q_s(t - La + a), \quad t \in [La, T_s + La - a].$$

Then for small a,

$$(3.24) \qquad \int_0^{T_s+La-a} (\frac{1}{2}|\dot{\xi}|^2 - V(\xi))dt - \int_0^{T_s}\left[\frac{1}{2}|\dot{Q}_s|^2 - V(Q_s)\right] dt$$
$$= \int_0^{La}\left[\frac{1}{2L^2}|\dot{Q}_s(\frac{t}{L})|^2 - V(Q_s(\frac{t}{L}))\right] dt$$
$$- \int_0^a \left[\frac{1}{2}|\dot{Q}_s(t)|^2 - V(Q_s(t))\right] dt$$
$$= \left(\frac{1}{2L^2}|\dot{Q}_s(0)|^2 + o(a)\right) La - \left(\frac{1}{2}|\dot{Q}_s(0)|^2 + o(a)\right) a < 0.$$

Introducing a function η_r as in the proof of (e) of Proposition 3.8 with ξ replacing Q_s, (3.24) shows

(3.25) $$\lim_{r\to\infty} I(\eta_r) < \inf_{\Gamma_i} I,$$

Contrary to (e).

It remains to prove (c) and (d). Thus suppose

(3.26) $$Q_s(0,T_s) \cap (\cup S_j) \neq \phi.$$

If $Q_s(\tau)$ lies in this intersection, $V(Q_s(\tau)) = 0$ while $V'(Q_s(\tau)) \neq 0$. It follows that the set of such τ's must be isolated. Suppose there are $m(s)$ such points: $0 < \tau_{1,s} < \cdots < \tau_{m(s),s} < T_s$. Set $T_{k,s} = t_{k,s} - t_{k-1,s}$ and

$$Q_{k,s}(t) = Q_s(t - \tau_{k-1,s}) : [0, T_{k,s}] \to \mathbf{R}^n, \ \ 1 \le k \le m(s).$$

Consider the new set of $\mu = \sum_{s=1}^{k+1} m(s)$ functions

$$P_1(t) = Q_{1,1}(t), \dots, P_{m(1)}(t) = Q_{m(1),1}(t), P_{1+m(1)}(t) = Q_{1,2}(t), \text{etc.}$$

on the associated t intervals. We claim $\mu \le \ell$. Otherwise for some $k < \hat{k}$, $P_k(\hat{T}_k)$ and $P_{\hat{k}}(\hat{T}_{\hat{k}}) \in \partial S_\rho$ for associated $\hat{T}_k, \hat{T}_{\hat{k}}$ and some ρ. Then as in the proof of (e), the functions $P_1, \dots, P_k, P_{\hat{k}+1}, \dots, P_\mu$ can be made part of a curve η_r such that as $r \to \infty$,

(3.27) $$\begin{aligned} I(\eta_r) \to & \sum_{\substack{s=1 \\ s \notin [k+1,\hat{k}] \cap \mathbf{N}}}^{\mu} \int_0^{\hat{T}_s} \left[\frac{1}{2}|\dot{P}_s|^2 - V(P_s)\right] dt \\ & < \sum_{s=1}^{\mu} \int_0^{\hat{T}_s} \left[\frac{1}{2}|\dot{P}_s|^2 - V(P_s)\right] dt, \end{aligned}$$

contrary to (e). Thus (c) and (d) hold and the proof of Proposition 3.8 and analysis of case (i) are complete.

Now we turn to:

Case (ii): $\mathcal{T}_\epsilon = \phi$ for a sequence $\epsilon_m \to 0$.

Then the arguments of Theorem 2.22, especially Lemmas 2.26 and 2.30, along a subsequence q_{ϵ_m} converges on C^2_{loc} to a solution q of (HS) with $q \in \Gamma_i$ and

(3.28) $$I(q) = \inf_{w \in \Gamma_i} I(w).$$

Hence q reflected about $t = 0$ provides the homoclinic orbit for this case.

Remark 3.29: For case (ii), it is possible that for some $\gamma > 0$, $q(\gamma) \in S_j$ for some $j \neq i$. However as in the construction of the σ's and τ's in (3.9), there can be at most $\ell - 1$ such points, each corresponding to a different set S_j. If V is C^2, there cannot exist any such points γ since the solution to the initial value problem for (HS) with $w(\gamma) = x \in S_j$ and $\dot{w}(\gamma) = 0$ would have as solutions both a homoclinic orbit joining x and 0 and a periodic orbit joining x and S_i.

Remark 3.30: The case (ii) follows in particular for any i such that

$$c_i = \min_{1 \leq j \leq \ell} c_j.$$

Completion of the proof of Theorem 3.1: It remains only to show that

$$\ell_h + \ell_p \geq \ell. \tag{3.31}$$

Let m be the number of chains of homoclinic and periodic solutions of energy 0 joining the set of S_j's to 0. We have shown for each S_i, there is at least one such chain. Hence $m \geq \ell$. On the other hand, it is easy to see that $m \leq \ell_h + \ell_p$ and (3.31) follows.

Remark 3.32: The hypothesis that $V'(x) \neq 0$ on $\partial\Omega$ can be weakened. See Remark 2.41. One still gets an analogue of (3.31) where heteroclinic orbits are also included provided that one has enough regularity, e.g. C^1, for $\partial\Omega$.

§4. $\Omega = \mathbf{R}^n$.

In this section, a variant of Theorem 2.1 will be proved for $\Omega = \mathbf{R}^n$. The existence of an orbit emanating from infinity and terminating at 0 will also be established.

Theorem 4.1: Suppose $V \in C^1(\mathbf{R}^n, \mathbf{R})$, $V(0) > V(x)$ for all $x \in \mathbf{R}^n \backslash \{0\}$, and

$$\overline{\lim}_{|x|\to\infty} V(x) < V(0). \tag{4.2}$$

Then for each $\xi \in \mathbf{R}^n \backslash \{0\}$, there is a solution $q = q(t)$ of (HS) such that $q(0) = \xi$ and $q(\infty) = 0 = \dot{q}(\infty)$.

Proof: Again, without loss of generality, assume $V(0) = 0$. Set

$$\Gamma_\xi = \{w \in E \mid w(0) = \xi \text{ and } w(\infty) = 0\} \tag{4.3}$$

and

$$c_\xi = \inf_{w \in \Gamma_\xi} I(w). \tag{4.4}$$

We claim c_ξ is a critical value of I and any corresponding critical point q_ξ is the desired solution of (HS). The proof parallels that of Theorem 2.1 but is a bit simpler so we will omit it.

Remark 4.5: A more general result than Theorem 4.1 has been obtained by Bolotin and Kozlov using arguments from Riemannian geometry. See e.g. [2, Theorem 10].

Remark 4.6: Theorem 4.1 shows there is a solution of (HS) of energy 0 joining each point in $\mathbf{R}^n$ to 0. Thus it is natural to ask whether there is a 0 energy orbit joining 0 and ∞. The next result establishes the existence of such an oribt emanating from ∞ and terminating at 0. Reversing time. we get an "escape orbit" of (HS). Since no assumptions have been made on the behavior of $V(x)$ as $|x| \to \infty$, this escape may occur in finite time. More precisely we have:

Theorem 4.7: Under the hypotheses of Theorem 4.1, there is a solution q of (HS) such that $q(\infty) = 0 = \dot{q}(\infty)$ and $|q(t)| \to \infty$ as $t \to T^*$ where $T^* = 0$ or $-\infty$.

Proof: We will find a solution q of (HS) such that $q(\infty) = 0$, $q(0) \in \partial B_1$. and $|q(t)| \to \infty$ as $t \to T$ for some $T \in [-\infty, 0)$. Rescaling time if $T > -\infty$ yields the statement of the Theorem.

Two simple preliminary results are needed for the proof of Theorem 4.7.

Lemma 4.8: Let Γ_ξ and c_ξ be as defined in (4.3)-(4.4). Suppose $q_\xi \in \Gamma_\xi$ and $I(q_\xi) = c_\xi$. Let $\tau \in (0, \infty)$ and $\eta = q_\xi(\tau)$. Then $w(t) = q_\xi(t+\tau) \in \Gamma_\eta$ and $I(w) = c_\eta$.

Proof: Clearly $w \in \Gamma_\eta$. If $I(w) > c_\eta = I(q_\eta)$, set

$$\begin{aligned} Q(t) &= q_\xi(t), t \in [0, \tau] \\ &= q_\eta(t - \tau), t \geq \tau \end{aligned}$$

then $Q \in \Gamma_\xi$ but

$$\begin{aligned} I(Q) &= \int_0^\tau \left[\frac{1}{2}|\dot{q}_\xi|^2 - V(q_\xi)\right] dt + c_\eta \\ &< \int_0^\tau \left[\frac{1}{2}|\dot{q}_\xi|^2 - V(q_\xi)\right] dt + \int_0^\infty \left[\frac{1}{2}|\dot{w}|^2 - V(w)\right] dt \\ &= I(q_\xi), \end{aligned} \tag{4.9}$$

contrary to (4.4).

Proposition 4.10: Let $\xi_k \to \xi$ in $\mathbf{R}^n$. Then there is a sequence $q_{\xi_k} \in \Gamma_{\xi_k}$ such that $I(q_{\xi_k}) = c_{\xi_k}$ and $q_{\xi_k} \to q_\xi$ in $C^2_{\text{loc}}(\mathbf{R}^+, \mathbf{R}^n)$ along a subsequence where $q_\xi \in \Gamma_\xi$ and $I(q_\xi) = c_\xi$.

Proof: Since (ξ_k) is a bounded sequence, it is easily verified via (4.4) that the critical values c_{ξ_k} are bounded, e.g. by M, independently of k. It then follows from Lemma 2.9 that the functions q_{ξ_k} are bounded in $L^\infty(\mathbf{R}^+, \mathbf{R}^n)$. Hence (HS) implies (q_{ξ_k}) is bounded in $C^2(\mathbf{R}^+, \mathbf{R}^n)$. Consequently, as earlier, a subsequence of q_{ξ_k} converges in C^2_{loc} to a solution q of (HS) such that $q(0) = \xi$. Moreover $I(q_{\xi_k}) \leq M$ implies $I(q) \leq M$. The form of I shows $q \in E$ and a slight variation of Proposition 2.11 gives $q(\infty) = 0$. Thus $q \in \Gamma_\xi$.

It remains to prove that $I(q) = c_\xi$. If not, by (4.4), $I(q) > c_\xi$. Choose α so that

$$I(q) > \alpha > c_\xi. \tag{4.11}$$

Hence there is a $T > 0$ such that

$$\int_0^T \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt > \alpha > c_\xi. \tag{4.12}$$

Since $q_{\xi_k} \to q_\xi$ in C^2_{loc} along a subsequence, for large such k,

$$\int_0^T \left[\frac{1}{2}|\dot{q}_{\xi_k}|^2 - V(q_{\xi_k})\right] dt > \alpha > c_\xi. \tag{4.13}$$

Therefore

$$c_{\xi_k} = I(q_{\xi_k}) > \alpha > c_\xi \tag{4.14}$$

for such k. On the other hand, suppose $w \in \Gamma_\xi$ and $I(w) = c_\xi$. Let $\epsilon > 0$ and set

$$\begin{aligned} Q_k(t) &= \left(1 - \tfrac{t}{\epsilon}\right)\xi_k + \tfrac{t}{\epsilon}\xi && t \in [0, \epsilon] \\ &= w(t - \epsilon) && t > \epsilon. \end{aligned} \tag{4.15}$$

Then $Q_k \in \Gamma_{\xi_k}$ and

$$c_{\xi_k} \le I(Q_k) = \frac{1}{2\epsilon}|\xi_k - \xi|^2 - \int_0^\epsilon V(Q_k(t))dt + c_\xi. \tag{4.16}$$

Thus (4.15)-(4.16) show

$$\overline{\lim}_{k\to\infty} c_{\xi_k} \le -\epsilon V(\xi) + c_\xi. \tag{4.17}$$

Since ϵ is arbitrary, (4.17) is contrary to (4.14). Hence $c_\xi = I(q)$.

Now we are ready for the:

Proof of Theorem 4.7: Let $(x_k) \subset \mathbf{R}^n$ satisfy $|x_k| \to \infty$ as $k \to \infty$. For each k, there is a solution $q_k \equiv q_{x_k}$ of (HS) given by Theorem 4.1. Since $q_k(\infty) = 0$, there is a $\tau_k > 0$ such that $\xi_k \equiv q_k(\tau_k) \in \partial B_1(0)$ and $q_k(t) \in B_1(0)$ for all $t > \tau_k$. Set $\tilde{q}_k(t) = q_k(t + \tau_k)$. Therefore $\tilde{q}_k(0) \in \partial B_1$ and $\tilde{q}_k$ is a solution of (HS). In fact, by Lemma 4.8, it can be assumed that $\tilde{q}_k = q_{\xi_k}$. Since $(\xi_k) \subset \partial B_1$, by choosing a subsequence if necessary, we have $\xi_k \to \xi \in \partial B_1$. Consequently by Proposition 4.10, a subsequence of $\tilde{q}_k$ converges in C^2_{loc} to $q \in \Gamma_\xi$ with $I(q) = c_\xi$. This function q is defined for $t \in \mathbf{R}^+$. We will show there is a $T < 0$ such that q extends to a solution of (HS) in such a way that it satisfies the conclusions of Theorem 4.7.

By the choice of (x_k), there is a $\tau_{k,1} < \tau_k$ such that $q_k(\tau_{k,1}) \in \partial B_2$ and $q_k(t) \in B_2$ for all $t > \tau_{k,1}$. Consider the sequence $(\tau_k - \tau_{k,1})$. We claim this sequence is bounded. If not, $\tau_k - \tau_{k,1} \to \infty$ along a subsequence. Clearly

$$\begin{aligned} \int_{\tau_{k,1}}^{\tau_k} \left[\frac{1}{2}|\dot{q}_k|^2 - V(q_k)\right] dt &\le c_{q_k(\tau_{k,1})} \\ &\le \sup\{c_\eta \mid \eta \in \partial B_2\} < \infty. \end{aligned} \tag{4.18}$$

Since $-V$ is bounded away from 0 in any deleted neighborhood of 0, (4.18) shows for any $\epsilon > 0$ and for all large k, the orbit q_k must enter B_ϵ. Therefore there is an $s_k \in (\tau_{k,1}, \tau_k)$ such that $q_k(s_k) \in B_\epsilon$. Set

$$\begin{aligned} Q_k(t) &= q_k(t), && t \in [0, s_k] \\ &= (s_k + 1 - t)q_k(s_k), && t \in [s_k, s_k + 1] \\ &= 0, && t > s_k + 1 \end{aligned} \tag{4.19}$$

Then $Q_k \in \Gamma_{x_k}$ and

$$I(Q_k) - I(q_k) = \frac{1}{2}|q_k(s_k)|^2 - \int_{s_k}^{s_k+1} V(Q_k(t))dt - \int_{s_k}^{\infty} \left[\frac{1}{2}|\dot{q}_k|^2 - V(q_k)\right] dt. \tag{4.20}$$

The first two terms on the right hand side of (4.20) go to 0 as $\epsilon \to 0$. The curve q_k joins B_ϵ to $B_2 \backslash B_1$ to 0 as t varies between s_k and ∞. Hence

$$\int_{s_k}^{\infty} \left[\frac{1}{2}|\dot{q}_k|^2 - V(q_k)\right] dt \tag{4.21}$$

has a positive lower bound independent of ϵ via Lemma 2.9. But then $I(Q_k) < I(q_k)$, contrary to the choice of q_k. It follows that $(\tau_k - \tau_{k,1})$ is a bounded sequence. Hence along a subsequence $\tau_k - \tau_{k,1} \to t_1 > 0$.

By appropriately rescaling time intervals again, a subsequence of $\tilde{q}_k$ (or q_k) converges to an extension of q (as a solution of (HS)) on $[-t_1, 0]$ such that $q(-t_1) \in \partial B_2$ and $q(t) \in B_2$ for $t \in (-t_1, 0)$. Continuing this construction yields an extension of q as a solution of (HS) to a maximal interval (T, ∞). Moreover there exists an increasing sequence t_k such that $q(-t_k) \in \partial B_{k+1}$ and $q(t) \in B_{k+1}$ for $t > -t_k$. This implies that

$$T = \inf_{k \in \mathbf{N}} \{-t_k\}$$

and $q(t)$ is not bounded as $t \to T$.

It remains to prove that $|q(t)| \to \infty$ as $t \to T$. Suppose that this is not the case. Then there is a $\mu > 0$ and sequences $s_k \to T,\ \sigma_k \to T$ such that

$$s_1 > \sigma_1 > s_2 > \sigma_2 > \cdots,$$

$|q(s_k)| = \mu$, $|q(\sigma_k)| = \mu + 1$, and $\mu < |q(t)| < \mu + 1$ for $t \in (\sigma_k, s_k)$. Now along a subsequence,

$$q(s_k) = \lim_{j \to \infty} \tilde{q}_j(s_k) \tag{4.22}$$

and by Proposition 4.10,

$$\int_{s_k}^{\infty} \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt = \inf_{w \in \Gamma_{q(s_k)}} I(w) \le \le \max\{c_\eta \mid \eta \in \partial B_\mu(0)\} \le M^*. \tag{4.23}$$

It is clear that M^* is bounded independently of k. Hence letting $k \to \infty$,

$$\int_T^\infty \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt \le M^*. \tag{4.24}$$

On the other hand, let

$$\beta_1 = \min_{\mu \le |x| \le \mu+1} -V(x) \tag{4.25}$$

Then by Lemma 2.9,

$$\int_{\sigma_k}^{\tau_k} \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt \ge \sqrt{2\beta_1}|q(\tau_k) - q(\sigma_k)| \ge \sqrt{2\beta_1} \tag{4.26}$$

so

$$\int_T^\infty \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt = \infty \tag{4.27}$$

contrary to (4.24). Thus $|q(t)| \to \infty$ as $t \to T$ and the proof is complete.

For our final result in this section we will given an analogue of Theorem 4.7 for a potential V which is singular.

Theorem 4.28: Suppose there is an $x_0 \in \mathbf{R}^n \backslash \{0\}$ and $V \in C^1(\mathbf{R}^n \backslash \{x_0\}, \mathbf{R})$ such that $V(0) = 0$, $V(x) < 0$ for $x \in \mathbf{R}^n \setminus \{0, x_0\}$

$$\overline{\lim}_{|x| \to \infty} V(x) < 0, \tag{4.29}$$

and $V(x) \to -\infty$ as $x \to x_0$. Then there is a solution, q, of (HS) such that $q(\infty) = 0$ and $q(t) \to x_0$ as $t \to T^*$ where $T^* = 0$ or $-\infty$.

Proof: Since the proof follows lines previously explored, we will only sketch the proof. Let $\epsilon_k \to 0$. Let $\Omega_k = \mathbf{R}^n \backslash B_{\epsilon_k}(x_0)$. Then by a combination of the proofs of Theorems 2.1 and 4.1, there is an $x_k \in \partial B_{\epsilon_k}(x_0)$ and a solution q_k of (HS) such that $q_k(0) = x_k$, $q_k(\infty) = 0$, and $q_k(t) \in \Omega_k$ for $t \in (0, \infty)$. Let $\rho \in (0, |x_0|)$. Hence there is a $\tau_k > 0$ such that $\xi_k \equiv q_k(\tau_k) \in \partial B_\rho$ and $q_k(t) \in B_\rho$ for all $t > \tau_k$. Now continuing as in the proof of Theorem 4.7, we get q as a limit of a subsequence of (q_k) after rescaling time.

§5. Some time dependent cases.

In this section the existence of homoclinic orbits for some time-dependent Hamiltonian systems will be studied. Consider

$$\ddot{q} + V_q(t, q) = 0 \tag{5.1}$$

where V satisfies

(V_1) $V(t, x) = -\frac{1}{2}L(t)x \cdot x + W(t, x)$,

(V_2) $L(t) \in C(\mathbf{R}, \mathbf{R}^{n^2})$ is a positive definite symmetric matrix for all $t \in \mathbf{R}$,

(V_3) $W \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ and there is a constant $\mu > 2$ such that

$$0 < \mu W(t, x) \leq x \cdot W_q(t, x)$$

for all $x \in \mathbf{R}^n \backslash \{0\}$ and $t \in \mathbf{R}$,

(V_4) $W_q(t, x) = o(|x|)$ as $x \to 0$ uniformly in $t \in \mathbf{R}$.

Note that (V_3)-(V_4) imply

$$W(t, x) = o(|x|^2) \tag{5.2}$$

as $x \to 0$, uniformly for $t \in \mathbf{R}$. Hence by (V_1)-(V_4), $x = 0$ is a local maximum for all $t \in \mathbf{R}$. However it is not a global maximum since by (V_3) for each $t \in \mathbf{R}$ is an $a_1(t)$ such that

$$W(t, x) \geq a_1(t)|x|^\mu \quad \text{for large} \quad |x|. \tag{5.3}$$

We do not know if simple minimization arguments in the spirit of §2-4 can be used to treat (5.1). In [5] assuming (V_1)-(V_4), and that L and W are T-periodic in t, it was shown that (5.1) possesses a homoclinic orbit emanating from 0. Analogous results for general Hamiltonian systems were obtained by Coti-Zelati, Ekeland and Sere [3] and Hofer and Wysocki [4]. We study (5.1) without periodicity assumptions on L and W. Two results will be obtained. The first is:

Theorem 5.4: Suppose V satisfies (V_1)-(V_4) and

(V_5) The smallest eigenvalue of $L(t) \to \infty$ as $|t| \to \infty$, ie.

$$\inf_{|\xi|=1} L(t)\xi \cdot \xi \to \infty \quad \text{as } |t| \to \infty,$$

(V_6) There is a $\overline{W} \in C(\mathbf{R}^n, \mathbf{R})$ such that

$$|W(t, x)| + |W_q(t, x)| \leq |\overline{W}(x)|$$

for all $x \in \mathbf{R}^n, t \in \mathbf{R}$.

Then there exists a (nontrivial) homoclinic orbit q of (5.1) emanating from 0 and such that

$$0 < \int_{-\infty}^{\infty} \left[\frac{1}{2}|\dot{q}|^2 - V(t, q)\right] dt < \infty$$

Proof: Let

$$\tilde{E} = \{q \in W^{1,2}(\mathbf{R}, \mathbf{R}^n) \mid \int_{-\infty}^{\infty} \left[|\dot{q}|^2 + L(t)q \cdot q\right] dt < \infty\}.$$

Then $\tilde{E}$ is a Hilbert space and as norm in $\tilde{E}$ we can take

(5.5) $$\|q\|^2 = \int_{-\infty}^{\infty} [|\dot{q}|^2 + L(t)q \cdot q] dt.$$

Note that

(5.6) $$\tilde{E} \subset W^{1,2}(\mathbf{R}, \mathbf{R}^n) \subset L^p(\mathbf{R}, \mathbf{R}^n)$$

for all $p \in [2, \infty]$ with the embedding being continuous. For $q \in \tilde{E}$ let

(5.7) $$I(q) = \frac{1}{2}\|q\|^2 - \int_{-\infty}^{\infty} W(t, q) dt.$$

Then $I \in C^1(\tilde{E}, \mathbf{R})$ and it is routine to verify that any critical point of I on $\tilde{E}$ is a classical solution of (5.1) with $q(\pm\infty) = 0 = \dot{q}(\pm\infty)$. See e.g. [1]. To establish the existence of a critical point of I, a variant of a standard "Mountain Pass" argument will be employed. The usual Mountain Pass Theorem does not apply here since the Palais-Smale condition does not hold due to the fact that we are working with functions on the unbounded set $\mathbf{R}$. However by (5.7) and (5.2), there are constants $\alpha, \rho > 0$ such that

(5.8) $$I(q) \geq \alpha$$

for $\|q\| = \rho$. Moreover by (V_3) – see e.g. [5] – there is a $q_0 \in \tilde{E}$ such that $\|q_0\| > \rho$ and

(5.9) $$I(q_0) < 0 = I(0).$$

Set

$$\mathcal{K} = \{g \in C([0, 1], \tilde{E}) \mid g(0) = 0 \text{ and } g(1) = q_0\}$$

and

(5.10) $$c = \inf_{g \in \mathcal{K}} \max_{s \in [0,1]} I(g(s)).$$

By (5.8)-(5.10)

$$c \geq \alpha. \tag{5.11}$$

Now Ekeland's Variational Principle – see e.g. Theorem 4.10 and 4.3 in [6] – implies there is a sequence $(q_k) \subset \tilde{E}$ such that:

$$I(q_k) \to c \quad \text{and} \quad I'(q_k) \to 0 \tag{5.12}$$

as $k \to \infty$. We will show a subsequence of q_k converges to a critical point, q of I with $q \not\equiv 0$. Hence by earlier remarks q is a solution of (5.1) of the desired type.

As a first step, note that (q_k) is a bounded sequence. Indeed by (V_3) and (5.12), for large k,

$$\begin{aligned} c + 1 + \|q_k\| &\geq I(q_k) - \frac{1}{\mu} I'(q_k) q_k \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|q_k\|^2 - \int_{-\infty}^{\infty} [W(t, q_k) - \frac{1}{\mu} W_q(t, q_k) \cdot q_k] dt \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|q_k\|^2 \end{aligned} \tag{5.13}$$

from which the result follows.

Since (q_k) is bounded, it possesses a weakly convergent subsequence in $\tilde{E}$. Let q denote its weak limit. Note that q_k converges to q in $L^{\infty}_{\text{loc}}(\mathbf{R}, \mathbf{R}^n)$. Hence $I'(q_k) \to 0$ easily implies $I'(q) = 0$, i.e. q is a critical point of I. It remains only to prove that $q \not\equiv 0$. Since $q_k \to q$ in L^{∞}_{loc}, $q_k \to q$ in $L^2([-A, A], \mathbf{R}^n)$ for all $A < \infty$. Hence it suffices to show there is an $A > 0$ such that $q_k \not\to 0$ in $L^2([-A, A], \mathbf{R}^n)$.

Proposition 5.14: *If $q_k \to 0$ in $L^2([-A, A], \mathbf{R}^n)$, there exists an $a > 0$ and independent of A such that*

$$\overline{\lim}_{k \to \infty} \|q_k\|^2_{L^2(\mathbf{R}, \mathbf{R}^n)} \leq \frac{a}{\beta(A)} \tag{5.15}$$

where

$$\beta(A) = \inf_{|\xi| = 1, |t| \geq A} L(t) \xi \cdot \xi.$$

Proof. Set $a = \sup_k \|q_k\|^2$. Consequently

$$\begin{aligned} \|q_k\|^2_{L^2(\mathbf{R}, \mathbf{R}^n)} &= \int_{-A}^{A} |q_k|^2 dt + \int_{\mathbf{R} \setminus [-A, A]} |q_k|^2 dt \\ &\leq \int_{-A}^{A} |q_k|^2 dt + \frac{1}{\beta(A)} \int_{\mathbf{R} \setminus [-A, A]} L(t) q_k \cdot q_k dt \\ &\leq \int_{-A}^{A} |q_k|^2 dt + \frac{a}{\beta(A)}. \end{aligned} \tag{5.16}$$

Letting $k \to \infty$, (5.15) is immediate.

To complete the proof of Theorem 5.4, we show that for A sufficiently large, $q_k \not\to 0$ in $L^2([-A, A], \mathbf{R}^n)$. Indeed by (5.12) and the boundedness of (q_k),

$$I(q_k) - \frac{1}{2}I'(q_k)q_k = \tag{5.17}$$
$$= \int_{-\infty}^{\infty} \left[\frac{1}{2}W_q(t, q_k) \cdot q_k - W(t, q_k)\right] dt \to c > 0$$

as $k \to \infty$. By (V_4) and (V_6) for each $M > 0$, there exists a $K_M > 0$ such that

$$|\frac{1}{2}W_q(t, x) \cdot x - W(t, x)| \le K_M|x|^2 \tag{5.18}$$

for all $|x| \le M$ and $t \in \mathbf{R}$. Taking

$$M = \sup_k \|q_k\|_{L^\infty},$$

by Proposition 5.14,

$$c = \lim_{k\to\infty} \int_{-\infty}^{\infty} \left[\frac{1}{2}W_q(t, q_k) \cdot q_k - W(t, q_k)\right] dt \tag{5.19}$$
$$\le K_M \overline{\lim}_{k\to\infty} \|q_k\|^2_{L^2(\mathbf{R},\mathbf{R}^n)} \le \frac{K_M a}{\beta(A)}.$$

By (V_5), $\beta(A) \to \infty$ as $A \to \infty$. But then (5.19) contradicts (5.11). The proof is complete.

Remark 5.20: By (5.11) and (5.19), the above argument works whenever there is an A such that

$$\beta(A) > \frac{K_M a}{\alpha}. \tag{5.21}$$

More careful estimates than those given above show that (V_5) can be weakened and (5.21) still obtains if $\beta(A)$ is large enough relative to

$$\underline{L} \equiv \inf_{t\in\mathbf{R}} \sup_{|\xi|=1} L(t)\xi \cdot \xi. \tag{5.22}$$

Indeed (5.13) provides an upper bound for $\|q_k\|$ depending only on μ and c. We assume $\underline{L} \le \sup_{|\xi|=1} L(t)\xi \cdot \xi \le 2\underline{L}$ on the interval $[-h, h]$. Note that we may assume $\operatorname{supp} q_0 \subset [-h, h]$ without loss of generality. Let $g(s) = sq_0 \in \mathcal{K}$. Then

$$\overline{c} = \max_{s\in[0,1]} I(g(s))$$

is an upper bound for c depending only on $\underline{L}$. Since there is a constant $b > 0$ depending only on $\underline{L}$ such that

$$\|w\|_{L^\infty(\mathbf{R},\mathbf{R}^n)} \leq b\|w\|$$

for all $w \in \tilde{E}$, the upper bound for $\|q_k\|$ gives an upper bound for $\|q_k\|_{L^\infty}$ and therefore for M. We can also easily see that $\alpha > 0$ depends only on $\underline{L}$. Thus we see the right hand side of (5.21) depends only on $\underline{L}$. Therefore (5.21) holds and Theorem 5.4 remains valid if $\beta(A)$ is large enough relative to $\underline{L}$.

Our final result is a variant of Theorem 5.4:

Theorem 5.23: Suppose V satisfies (V_1)-(V_4),

(V_7) There is a T periodic function L_∞ satisfying (V_2) such that

(i) $L_\infty(t) \geq L(t)$ for all $t \in \mathbf{R}$ i.e.

$L(t)x \cdot x \geq L(t)x \cdot x$ for all $t \in \mathbf{R}$ and $x \in S^{n-1}$,

(ii) $L_\infty(\tau)x \cdot x > L(\tau)x \cdot x$ for some $\tau \in \mathbf{R}$ and all $x \in S^{n-1}$,

(iii) $|L(t) - L_\infty(t)| \to 0$ as $|t| \to \infty$.

(V_8) $W(t,x)$ is T-periodic in t.

(V_9) the map $s \to s^{-1}W_q(t, sx) \cdot x$ is a strictly increasing function of $s \in (0, \infty)$ for all $t \in \mathbf{R}$ and $x \in \mathbf{R}^n \setminus \{0\}$.

Then there exists a nontrivial homoclinic orbit of (5.1) emanating from 0.

Remark 5.24: Condition (V_9) is related to hypotheses that have been used by Nehari [7-8], Coffman [9], Hempel [10], and others. We suspect that Theorem 5.23 is true without this hypothesis.

Proof of Theorem 5.23: The proof follows the same lines as that of Theorem 5.4. Now we work in $E_1 \equiv W^{1,2}(\mathbf{R}, \mathbf{R}^n)$. Note that by (V_2) and (V_7),

$$\left(\int_{-\infty}^{\infty} (|\dot{q}|^2 + L(t)q \cdot q)dt\right)^{1/2}$$

and

$$\left(\int_{-\infty}^{\infty} (|\dot{q}|^2 + L_\infty(t)q \cdot q)dt\right)^{1/2}$$

are equivalent norms in E_1. Hence I as defined in (5.7) belongs to $C^1(E_1, \mathbf{R})$. Moreover (5.8)-(5.9) are still valid in E_1. Defining c by (5.10), with $\tilde{E}$ replaced by E_1 in K, (5.11)-(5.13) still hold and $q_k \to q$ weakly in E_1 (along a subsequence) where $I'(q) = 0$. It remains only to prove that $q \not\equiv 0$. This will take some work.

The idea of the proof is to consider the functional

$$I^\infty(q) = \int_{-\infty}^{\infty} \left[\frac{1}{2}|\dot{q}|^2 + \frac{1}{2} L_\infty(t) q \cdot q - W(t, q) \right] dt. \tag{5.25}$$

By (V_7), (V_3), and (V_4), $I^\infty \in C^1(E_1, \mathbf{R})$ and by earlier remarks, critical points of I^∞ are homoclinic solutions of

$$\ddot{q} - L_\infty(t) q + W_q(t, q) = 0 \tag{5.26}$$

which emanate from 0. We will show: (a) I^∞ also has a "mountain pass" critical value c^∞. Moreover by (V_9), c^∞ can also be characterized as: (b)

$$c^\infty = \inf\{ I^\infty(y) \mid y \in E_1 \backslash \{0\} \text{ and } (I^\infty)'(y) = 0 \}.$$

This enables us to prove: (c) $c < c^\infty$. Then: (d) an argument related to the one following Proposition 5.14 implies $q \not\equiv 0$.

The steps (a)-(d) will now be carried out in detail.

Step (a): There exists a nontrivial homoclinic solution of (5.26) emanating from 0.

Proof: The existence of such a solution was established in [5]. However we will give another proof here since we need a minimax characterization of its corresponding critical value. Note that the constants $\rho, \alpha \geq 0$ and $q_0 \in E_1$ can be chosen so that the estimates (5.8)-(5.9) hold for both I and I^∞. Setting

$$c^\infty = \inf_{g \in K} \max_{s \in [0,1]} I^\infty(g(s)), \tag{5.27}$$

as in the proof of Theorem 5.4, there is a bounded sequence $(w_k) \subset E_1$ such that

$$I^\infty(w_k) \to c^\infty, \quad (I^\infty)'(w_k) \to 0. \tag{5.28}$$

Moreover a subsequence of w_k converges weakly in E_1 and also in L^∞_{loc} to $w \in E_1$ such that $(I^\infty)'(w) = 0$. Hence once we show $w \not\equiv 0$, as earlier it is a nontrivial homoclinic solution of (5.26) emanating from 0. Note that since L_∞ and W are T-periodic in t,

$$I^\infty(y(t)) = I^\infty(y(t + \ell T)) \tag{5.29}$$

for all $y \in E_1$ and $\ell \in \mathbf{Z}$. The functions $w_k \in E_1$. Hence $w_k(t) \to 0$ as $|t| \to \infty$ – see e.g. (20)-(24) in [5]. Consequenctly by (5.29), without loss of generality it can be assume that

w_k achieves its maximum in $[0, T]$. Thus if $w_k \to 0$ weakly in E_1 and in L^∞_{loc} as $k \to \infty$, in fact

$$w_k \to 0 \text{ in } L^\infty(\mathbf{R}, \mathbf{R}^n) \tag{5.30}$$

as $k \to \infty$. By (5.2),

$$\left|\frac{1}{2} W_q(t,x) \cdot x - W(t,x)\right| = o(1)|x|^2 \tag{5.31}$$

as $|x| \to 0$ uniformly for $t \in \mathbf{R}$. Hence by (5.30)-(5.31),

$$\int_{-\infty}^{\infty} \left[\frac{1}{2} W_q(t, w_k) \cdot w_k - W(t, w_k)\right] dt = o(1)\|w_k\|^2_{L^2(\mathbf{R},\mathbf{R}^n)} \to 0 \tag{5.32}$$

as $k \to \infty$. On the other hand by (5.28),

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[\frac{1}{2} W_q(t, w_k) \cdot w_k - W(t, w_k)\right] dt \\ &\quad = I^\infty(w_k) - \frac{1}{2}(I^\infty)'(w_k)w_k \to c^\infty \geq \alpha \end{aligned} \tag{5.33}$$

as $k \to \infty$, contrary to (5.32). Hence Step (a) is complete.

Remark 5.34: (5.28), (5.33), Fatou's Lemma and $(I^\infty)'(w) = 0$ imply (along our subsequence)

$$\begin{aligned} c^\infty &= \lim_{k\to\infty} \int_{-\infty}^{\infty} \left[\frac{1}{2} W_q(t, w_k) \cdot w_k - W(t, w_k)\right] dt \\ &\geq \int_{-\infty}^{\infty} \left[\frac{1}{2} W_q(t, w) \cdot w - W(t, w)\right] dt = I^\infty(w). \end{aligned}$$

Step (b): $c^\infty = \inf\{I^\infty(y) \mid y \in E_1\backslash\{0\},\ (I^\infty)'(y) = 0\}$.

Proof: Let $b = \inf\{I^\infty(y) \mid y \in E_1\backslash\{0\},\ (I^\infty)'(y) = 0\}$. By Remark 5.34,

$$c^\infty \geq b. \tag{5.35}$$

We claim for all $u \in E_1\backslash\{0\}$ such that $(I^\infty)'(u) = 0$, there is a $g \in \mathcal{K}$ such that

$$\max_{s\in[0,1]} I^\infty(g(s)) = I^\infty(u). \tag{5.36}$$

Then by (5.27),

$$c^\infty \leq b \tag{5.37}$$

so (5.35) and (5.37) yield Step (b).

To construct g, note that by (V_9), if $y \in E_1 \backslash \{0\}$, $I^\infty(sy)$ achieves its maximum along the ray $\{sy \mid s \geq 0\}$ at a unique point $s_0 y$ characterized by $(I^\infty)'(s_0 y)y = 0$. (See e.g. [9], [10].) Hence let g consist of a segment of the ray through 0 and u, a circular arc through $Ru/\|u\|$ and $Rq_0/\|q_0\|$, and a ray segment joining $Rq_0/\|q_0\|$ and q_0. Explicitly

$$g_R(s) = \begin{cases} 3sR\frac{u}{\|u\|} & s \in [0, \frac{1}{3}], \\ \frac{Ru}{\|u\|} \cos \frac{3\pi}{2}(s - \frac{1}{3}) + \frac{Rq_0}{\|q_0\|} \sin \frac{3\pi}{2}(s - \frac{1}{3}) & s \in [\frac{1}{3}, \frac{2}{3}], \\ 3(1-s)\frac{Rq_0}{\|q_0\|} + 3(s - \frac{2}{3})q_0 & s \in [\frac{2}{3}, 1]. \end{cases} \tag{5.38}$$

Then $g_R \in \mathcal{K}$ for all $R > 0$ and for R large, $I^\infty(g_R(s)) < 0$ for all $s \in [\frac{1}{3}, 1]$. Hence (5.36) holds and step (b) is verified.

Step (c): $c < c^\infty$.

Proof: Let g be the path constructed in Step (b) with $u = w$, w being as determined in Step (a). By (V_7) (i), for all $s \in [0, 1]$,

$$I(g(s)) \leq I^\infty(g(s)). \tag{5.39}$$

Hence

$$c \leq \max_{s \in [0,1]} I(g(s)) \leq I^\infty(w) = c^\infty. \tag{5.40}$$

If $c = c^\infty$, there is an $\overline{s} \in (0, 1)$ such that

$$c = I(g(\overline{s})) = I^\infty(g(\overline{s})) = c^\infty. \tag{5.41}$$

But there is a unique $s \in (0, 1)$ such that $I^\infty(g(s)) = c^\infty$, namely $s = \|w\|/3R$ and $g(s) = w$. Hence to prove that $c < c^\infty$, it suffices to show that

$$I(w) < I^\infty(w). \tag{5.42}$$

If $I(w) = I^\infty(w)$, by (V_7) (ii), $w(t) \equiv 0$ in a neighborhood of $t = \tau$. Observe that w is a solution of the linear system of equations

$$\ddot{q}_i + \sum_{j=1}^n a_{ij}(t) q_j = 0, \qquad 1 \leq i \leq n \tag{5.43}$$

where $a_{ij}(t) = 0$ if $w(t) = 0$ and if $w(t) \neq 0$,

$$a_{ij}(t) = \frac{1}{|w(t)|^2} \frac{\partial V}{\partial q_i}(t, w(t)) w_j(t).$$

The coefficients a_{ij} are continuous via (5.2). Since $w(t) \equiv 0$ near $t = \tau$ and satisfies a linear system with continuous coefficients, $w(t) \equiv 0$, a contradication. Thus (5.42) follows and Step (c) is proved.

Step (d). Completion of the proof of Theorem 5.23.

Recall that (q_k) is the "mountain pass" sequence for this problem satisfying (5.12) and q is the limit (along a subsequence) of q_k. We must show $q \not\equiv 0$. Due to the L^∞_{loc} convergence of q_k to q, it suffices to prove for some $A > 0$,

$$\int_{-A}^{A} |q_k|^2 dt \not\to 0 \tag{5.44}$$

along a subsequence. We will use a "concentration compactness" type argument. Suppose

$$\int_{-A}^{A} |q_k|^2 dt \to 0 \tag{5.45}$$

as $k \to \infty$ for all $A > 0$. Consider

$$\beta_k = \sup_{\ell \in \mathbf{Z}} \int_0^T |q_k(t + \ell T)|^2 dt. \tag{5.46}$$

We claim

$$\varliminf_{k \to \infty} \beta_k \neq 0. \tag{5.47}$$

Indeed for any $u \in E_1$, $s, t \in [0, T]$, and $\ell \in \mathbf{Z}$, we have

$$\begin{aligned} |u(t + \ell T)|^2 &= |u(s + \ell T)|^2 + \int_s^t \frac{d}{dr}|u(r + \ell T)|^2 dr \\ &\le |u(s + \ell T)|^2 + 2\int_0^T |u(r + \ell T) \cdot \dot{u}(r + \ell T)| dr. \end{aligned} \tag{5.48}$$

Integrating (5.48) for s over $[0, T]$ and setting $u = q_k$ yields:

$$\begin{aligned} |q_k(t + \ell T)|^2 &\le \frac{1}{T}\int_0^T |q_k(s + \ell T)|^2 ds \\ &\quad + \frac{2}{T}\left(\int_0^T |q_k(s + \ell T)|^2 ds\right)^{1/2} \|\dot{q}_k\|_{L^2(\mathbf{R}, \mathbf{R}^n)} \\ &\le \frac{\beta_k}{T} + \frac{2}{T}\beta_k^{1/2} M \end{aligned} \tag{5.49}$$

where M is an upper bound for $\|q_k\|$. Since ℓ is arbitrary, if $\beta_k \to 0$, (5.49) shows

$$\lim_{k\to\infty} \|q_k\|_{L^\infty(\mathbf{R},\mathbf{R}^n)} = 0. \tag{5.50}$$

But then the argument of (5.30)-(5.33) implies $c = 0$, a contradiction. Hence (5.47) holds.

Comparing (5.47) to (5.45), we see there must exist a sequence $\ell_k \in \mathbf{Z}$ such that

$$\begin{cases} \text{(i) } |\ell_k| \to \infty \text{ as } k \to \infty \text{ and} \\ \text{(ii) } \int_0^T |q_k(t+\ell_k T)|^2 dt \geq \gamma > 0. \end{cases} \tag{5.51}$$

Set $\tilde{q}_k(t) = q_k(t+\ell_k T)$. By (V_7) (iii) and a familiar argument, there is a subsequence of $\tilde{q}_k$ which converges weakly in E_1 and also in L^∞_{loc} to $\tilde{q} \in E_1$, a homoclinic solution of (5.26). Moreover by (5.51) (ii), $\tilde{q}$ is a nontrivial solution of (5.26). As in Remark 5.34, using (V_8),

$$\begin{aligned} c &= \lim_{k\to\infty} \int_{-\infty}^{\infty} \left[\frac{1}{2}W_q(t, q_k)\cdot q_k - W(t, q_k)\right] dt \\ &= \lim_{k\to\infty} \int_{-\infty}^{\infty} \left[\frac{1}{2}W_q(t\cdot \tilde{q}_k)\cdot \tilde{q}_k - W(t, \tilde{q}_k)\right] dt \\ &\geq \int_{-\infty}^{\infty} \left[\frac{1}{2}W_q(t, \tilde{q})\cdot \tilde{q} - W(t, \tilde{q})\right] dt = I^\infty(\tilde{q}). \end{aligned} \tag{5.52}$$

By Step (b),

$$I^\infty(\tilde{q}) \geq c^\infty. \tag{5.53}$$

But (5.52)-(5.53) imply $c \geq c^\infty$, contrary to Step (c). Hence (5.44) holds and $q \not\equiv 0$. The proof of Theorem 5.23 is complete.

REFERENCES

[1] Rabinowitz, P. H., Periodic and heteroclinic orbits for a periodic Hamiltonian system, to appear Analyse Nonlineaire.

[2] Kozlov, V. V., Calculus of variations in the large and classical mechanics, Russian Math. Surveys, **40**, (1985), 37-71.

[3] Coti-Zelati, V., I. Ekeland, and E. Sere, A variational approach to homoclinic orbits in Hamiltonian systems, preprint.

[4] Hofer, H. and K. Wysocki, First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, preprint.

[5] Rabinowitz, P. H., Homoclinic orbits for a class of Hamiltonian systems, to appear, Proc. Royal Soc. Edinburgh.

[6] Mawhin, J. and M. Willem, Critical Point Theory and Hamiltonian systems, Springer Verlag, 1989.

[7] Nehari, Z., On a class of nonlinear second order differential equations, Trans. Amer. Math. Soc. **95**, (1960), 101-123.

[8] Nehari, Z., Characteristic values associated with a class of nonlinear second-order differential equations, Acta. Math. **105**, (1961), 141-175.

[9] Coffman, C., A minimum-maximum principle for a class of nonlinear integral equations, J. Anal. Math. **22**, (1969), 391-419.

[10] Hempel, J. A., Superlinear boundary value problems and nonuniqueness, thesis, University of New England, Armidale, 1970.